

Problem Set 4

Chapter 3

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text.

- Z1. Let $T \in \mathcal{L}(V, W)$, and suppose that $\dim V = n$ and $\dim W = m$. From the derivation of $\dim V = \dim \text{null } T + \dim \text{range } T$ and what I talked about in class, there is a basis in V and a basis in W such that the matrix of T , $(a_{i,j})_{m \times n}$, has the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ m \end{array} \begin{pmatrix} & 1 & 2 & 3 & \dots & k & \dots & n \\ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{pmatrix}$$

A is an $m \times n$ matrix. There is a $k \times k$ identity matrix in the upper left corner. All other entries are 0. $k = \dim \text{range } T$.

- If $k < \min\{m, n\}$, is T surjective? Is T injective?
 - If $k = \min\{m, n\}$, is T surjective? Is T injective?
- Z2. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by $T(x, y) = (3x - y, y - x)$. What is the matrix for T using the standard basis in \mathbb{R}^2 ? What is the matrix for T using the basis $((1, 1), (1, -1))$? What is the “canonical matrix” for T ? By “canonical matrix”, I mean the form given in the first problem.
- Z3. Verify that $(1, 2, 2)$ is a solution to the equations:

$$\begin{aligned} 5x_1 - 2x_2 + 2x_3 &= 5 \\ 4x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 5, \end{aligned}$$

and then find all possible solutions. If there are no more, say so.

22. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

COMMENT: If ST is invertible, all you know is that there is some operator, R , such that $R(ST) = (ST)(R) = I$.

23. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Hint: Use the result of the previous problem.

24. Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Hint: This problem can be a bit tricky. There are many ways to solve it. The matrix equivalent is that if A is an $n \times n$ matrix then $AB = BA$ for all $n \times n$ matrices, B , if and only if A is of the form $A = (a_{i,j})_{n \times n}$, where $a_{i,i} = c$ and $a_{i,j} = 0$ if $i \neq j$. c is a scalar. Consider B matrices of the form $B_k = (b_{i,j}^k)$, where $b_{k,k}^k = 2$, $b_{i,i}^k = 1$ if $i \neq k$, and $b_{i,j}^k = 0$ if $i \neq j$. Consider the implications of $B_k A = A B_k$.

25. Prove that if V is finite dimensional with $\dim V > 1$, then the set of non-invertible operators on V is not a subspace of $\mathcal{L}(V)$.

You might wish to prove this in terms of matrices. No matter how you prove it, all you need is a single counterexample!

26. Not assigned. I believe that I dealt with this sufficiently in class. If you think otherwise, please let me know.