Elementary Linear Maps

Let $T \in \mathcal{L}(V, W)$, where $\dim V = n$ and $\dim W$. Furthermore, let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ be bases for V and T, respectively. Let $\mathcal{M}(T) = A$ with respect to the two bases. The scalars are $\mathbb{F}$.

For any i and j, $(1 \leq i \leq m$ and $1 \leq j \leq n)$, let $\Delta_{i,j}$ be the linear map defined by

$$\begin{array}{rcll} \Delta_{i,j}(v_k) & = & 0 & k \neq j \\ \Delta_{i,j}(v_j) & = & w_i & (k = j). \end{array}$$

$\dim \operatorname{null} \Delta_{i,j} = n - 1$, and $\operatorname{null} \Delta_{i,j}$ is spanned by all of the v_k vectors **except** for v_j, which is mapped to w_i and generates range $\Delta_{i,j}$.

Let $E_{i,j} = \mathcal{M}(\Delta_{i,j})$. Then row i and column j of $E_{i,j}$ contains 1. All other entries are 0.

The vector space of linear maps from V to W is isomorphic to the vector space of $m \times n$ matrices with entries in $\mathbb{F}$. In fact, the isomorphism can be defined by mapping $\Delta_{i,j}$ to $E_{i,j}$. These $m \times n$ linear operators (or matrices) are linearly independent and span $\{T : V \longrightarrow W \mid T \in \mathcal{L}(V, W)\}$ (or $\mathrm{Mat}(m, n, \mathbb{F})$).

Elementary linear operators

Now let $T \in \mathcal{L}(V)$, and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Define A, $\Delta_{i,j}$ and $E_{i,j}$ as above.

$\Delta_{i,j}T$ *versus* $T\Delta_{i,j}$

What are $\Delta_{i,j}T$ and $T\Delta_{i,j}$, and how do they compare?

$$\begin{array}{rcl} \Delta_{i,j}T(v_k) & = & \displaystyle\sum_{l=1}^{n} a_{l,k}\Delta_{i,j}(v_l) \\ & = & a_{j,k}v_i \end{array}$$

On the other hand,

$$\begin{aligned} T\Delta_{i,j}(v_k) &= T(0) \\ &= 0 \qquad \text{if } k \neq j, \text{ and} \\ T\Delta_{i,j}(v_j) &= T(v_i) \\ &= \sum_{l=1}^{n} a_{l,i} v_l \quad (k = j). \end{aligned}$$

$E_{i,j}A$ ***versus*** $AE_{i,j}$

What is the matrix equivalent? First note that

$$E_{i,j} = \begin{array}{c|cccccc} & 1 & 2 & \ldots & j & \ldots & n \\ \hline 1 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 2 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i & 0 & 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ n & 0 & 0 & \ldots & 0 & \ldots & 0 \end{array}.$$

Translating $\Delta_{i,j}T$ into matrix form yields

$$E_{i,j}A = \begin{array}{c|cccccc} & 1 & 2 & \ldots & j & \ldots & n \\ \hline 1 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 2 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i & a_{j,1} & a_{j,2} & \ldots & a_{j,j} & \ldots & a_{j,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ n & 0 & 0 & \ldots & 0 & \ldots & 0 \end{array},$$

and translating $T\Delta_{i,j}$ into matrix form yields

$$AE_{i,j} = \begin{array}{c|cccccc} & 1 & 2 & \ldots & j & \ldots & n \\ \hline 1 & 0 & 0 & \ldots & a_{1,i} & \ldots & 0 \\ 2 & 0 & 0 & \ldots & a_{2,i} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i & 0 & 0 & \ldots & a_{i,i} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ n & 0 & 0 & \ldots & a_{n,i} & \ldots & 0 \end{array}.$$

In words, multiplying A on the left with $\Delta_{i,j}$ creates a matrix whose i^{th} row is the j^{th} row of A. All other rows contain only 0s. Multiplying A on the right with $\Delta_{i,j}$ creates a matrix whose j^{th} column is the i^{th} column of A. All other columns contain only 0.

Commuting operators.

If $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$, then T and S are said to commute if $ST = TS$. It is clear that aI commutes with every operator, since $aIT = aT$ and $TaI = aTI = aT$. In matrix terms, the matrix aI contains a on the diagonal ($a_{1,1} = a_{2,2} = \cdots = a_{n,n} = a$) and zeros everywhere else ($a_{i,j} = 0$ if $i \neq j$.

Suppose that $T \in \mathcal{L}(V)$ and that T commutes with every operator on V. V has finite dimension, n. Then $T = aI$ for some scalar $a \in \mathbb{F}$.

Proof: If T commutes with every operator on V, then it must commute with $\Delta_{i,j}$ for all i and j between 1 and n. In matrix terms, if $A = \mathcal{M}(T)$, then $E_{i,j}A = AE_{i,j}$ for all i and j between 1 and n.

$E_{i,j}A$ contains a single row that is non-zero (i^{th} row) and $AE_{i,j}$ contains a single column that is non-zero (j^{th} column). If these matrices are equal, then $a_{j,k} = 0$ if $k \neq j$, $a_{l,i} = 0$ if $l \neq i$ and $a_{j,j} = a_{i,i}$ because the only possible non-zero term occurs in row i and column j of both matrices. The scalar in row i and column j of $E_{i,j}A$ is $a_{j,j}$ and that in row i and column j of $AE_{i,j}$ is $a_{i,i}$.

The above is true for any i and j, so $a_{i,j} = 0$ if $i \neq j$. Letting $a = a_{1,1}$, note that $a = a_{1,1} = a_{2,2} = \cdots = a_n$, so that $A = aI$.

The matrix proof is "more intuitive" in my opinion than a more abstract proof. Note that the choice of basis does not matter except that A and $E_{i,j}$ must be defined with respect to the same basis.