

Test 1 Due March 4, 2008

If $G = \mathbb{C}, \mathbb{R}, \mathbb{Q}$ or \mathbb{Z} , assume that the group operation is addition. Let $\mathbb{C}^*, \mathbb{R}^*$ and \mathbb{Q}^* refer to complex numbers without 0, real numbers without 0 and rational numbers without 0, respectively. Let \mathbb{R}^+ and \mathbb{Q}^+ refer to positive real numbers and positive rational numbers, respectively. Assume that the group operation in these cases (the last five) is multiplication (\times).

1. True or false. Give reasons.

- (a) In \mathbb{Z} , the function $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = 6n$ is a homomorphism.
- (b) In \mathbb{Z} , the function $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = 3n$ is an isomorphism.
- (c) In \mathbb{Z} , the function $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = -n$ is an isomorphism.
- (d) In \mathbb{Q}^* , the function $\phi : \mathbb{Q}^* \rightarrow \mathbb{Q}^*$ defined by $\phi(a) = 1/a$ is an isomorphism.
- (e) \mathbb{R} is isomorphic to \mathbb{R}^+ .

2. Let $G = \left\{ \begin{pmatrix} a & b \\ -7b & a \end{pmatrix} \mid a, b \in \mathbb{Q}, |a| + |b| > 0 \right\}$.

- (a) Show that G , with matrix multiplication as the binary relation, is an Abelian group.
- (b) Let $G' = \{a + b\sqrt{-7} \mid a, b \in \mathbb{Q}, |a| + |b| > 0\}$. Show that G' , with multiplication as the binary relation, is isomorphic to G .
- (c) Let $G_1 = \mathbb{Z} \pmod{5}$, with 0 removed. This is an Abelian group of order 4 under multiplication. Using trial and error, show that G_1 is cyclic.
- (d) Does $\sqrt{2}$ exist in G_1 ? That is, is there any $a \in G_1$ such that $a^2 = 2 \pmod{5}$?
- (e) Let $G_2 = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z} \pmod{5}, (a, b) \neq (0, 0)\}$. Show that G_2 is an Abelian group under "multiplication" defined by $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$. (All of this is $\pmod{5}$. That is, $7 - 11\sqrt{2} = 2 + 4\sqrt{2}$, for example.)
- (f) What is $|G_2|$?
- (g) Is G_2 cyclic?

3. (a) Compute $(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) / \langle (2, 3) \rangle$. Note that $\langle (2, 3) \rangle$ is the cyclic subgroup generated by $(2, 3)$.
- (b) What is the number of subgroups of $\mathbb{Z}/45\mathbb{Z}$?
- (c) Compute all the subgroups of $\mathbb{Z}/45\mathbb{Z}$ and draw a subgroup diagram as well.
- (d) Let p_1, p_2 and p_3 be three distinct prime numbers. How many distinct Abelian groups are there of order $p_1 p_2^2 p_3^3$ (up to isomorphism)?
- (e) Let $G = \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$. Compute all the subgroups of G .
- (f) Give an example of a finite Abelian group, G , that has a subgroup, H such that H is isomorphic to G/H .

Test1 continued

4. $15\mathbb{Z}$, $21\mathbb{Z}$ and $35\mathbb{Z}$ are three different subgroups of \mathbb{Z} . Compute the smallest subgroup of \mathbb{Z} that contains these three subgroups.

5. There are two types of “perfect” shuffles of a deck of (52) cards. a) Split the deck into two equal halves (26+26). Shuffle so that the bottom card in the shuffled deck is 26, the one on top of it is 52, the next is 25, and so on. The top card in the shuffled deck is 27. b) The deck is split the same way, but this time the bottom card is 52, the one on top of it is 26 and so on. The top card is 1. These shuffles generate permutations π_1 and π_2 of the deck, respectively.
 - (a) How many orbits are there in π_1 ? How many shuffles does it take for the deck to return to its original order?
 - (b) Same question as above for π_2 .